Chebyshev Expansions for Solutions of Linear Differential Equations

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Joint work with Bruno Salvy

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March 25, 2009

I Introduction

How to evaluate a function f in $[-1, 1]$?

Two representations of f:

- in Taylor series

$$f = \sum_{n=0}^{+\infty} c_n x^n, \; c_n = \frac{f^{(n)}(0)}{n!},$$

- or in Chebyshev series

$$f = \sum_{n=0}^{+\infty} t_n T_n(x),$$

$$t_n = \frac{1}{\pi} \int_{-1}^{1} T_n(t) \frac{f(t)}{\sqrt{1-t^2}} dt.$$

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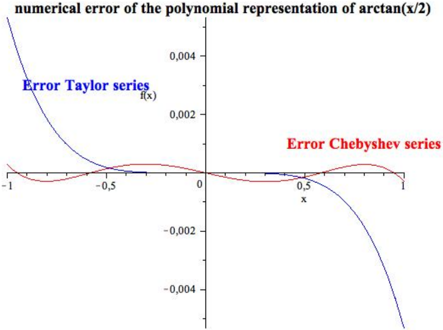

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Projects using Chebyshev series to represent functions in Matlab : Chebfun, Miscfun.

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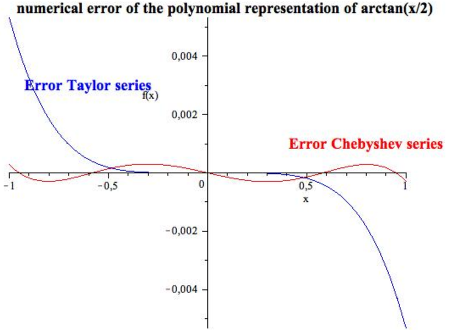

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How to compute t_n?

General case: numerical computation of the integral. Slow.

Computation of Coefficients with Recurrences

Theorem (60's)

If f is solution of a linear differential equation with polynomial coefficients, then the Chebyshev coefficients are cancelled by a linear recurrence with polynomial coefficients.

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Applications:

- Numerical computation of the coefficients.
- Computation of closed-form for coefficients.

Example ($f(x) = \arctan(x/2)$)

```
> def:=(4+x^2)*diff(y(x),x$2)+2*x*diff(y(x),x);
```

$$def := (4+x^2)\left(\frac{d^2}{dx^2}y(x)\right) + 2x\left(\frac{d}{dx}y(x)\right) \quad (15)$$

```
> rec:=Chebyshev:-diffeqtorecchebyshev(def,y(x),t(n))=0;
```

$$rec := n\,t(n) + (36+18n)\,t(n+2) + (n+4)\,t(n+4) = 0 \quad (16)$$

```
> r:=simplify(evalc(allvalues(rsolve({rec,seq(t(i)=1/Pi*int(arctan(x/2)*
T(i,x)/sqrt(1-x^2),x=-1..1),i=0..3)},t(n))))) assuming n::integer;
```

$$r := \begin{cases} 0 & n = 0 \\ \dfrac{\left(-2+\sqrt{5}\right)^{n+1}\left(\sqrt{5}+2\right)\sin\left(\frac{1}{2}n\pi\right)}{n} & \textit{otherwise} \end{cases} \quad (17)$$

State of the Art

- Clenshaw (1957): numerical scheme to compute the Chebyshev coefficients without computing all these integrals.
- Fox and Parker (1968): method for the computation of the Chebyshev recurrence relations for differential equations of small orders.
- Paszkowski (1975): algorithm for computing the Chebyshev recurrence relation.
- Lewanowicz (1976): algorithm for computing a smaller order Chebyshev recurrence relation in some cases.
- Rebillard (1998): new algorithm for computing the Chebyshev recurrence relation.

New Results (2009)

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- Implementation of algorithm in Maple.

II Fractions of Recurrence Operators

Morphisms of Rings of Operators ($S \cdot u_n = u_{n+1}$)

Taylor series ($f := \sum c_n x^n$)

$$xf = \sum c_n x^{n+1} = \sum c_{n-1} x^n,$$

$$f' = \sum n c_n x^{n-1} = \sum (n+1) c_{n+1} x^n$$

$$x \mapsto X := S^{-1},$$

$$\frac{d}{dx} \mapsto D := (n+1)S.$$

$$(4+x^2)\left(\frac{d}{dx}\right)^2 + 2x\frac{d}{dx}$$

$$\mapsto (4+S^{-2})(n+1)(n+2)S^2 + 2S^{-1}(n+1)S$$

$$= (n+1)\left(4(n+2)S^2 + n\right)$$

$$4(n+2)c_{n+2} + nc_n = 0$$

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Ore Polynomials: Framework for Recurrence Operators

- $\sum a_i(n)u_{n+i}$ is represented by $\sum a_i(n)S^i$.
- These polynomials are non-commutative.
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Example (gcld algorithm)

INPUT recurrence operators A and B
OUTPUT The "greatest" G such that
$A = G\tilde{A}$ and $B = G\tilde{B}$

Example (lclm algorithm)

INPUT recurrence operators A and B
OUTPUT U and V such that
$UA = VB$

Fractions of Recurrence Operators (Ore 1933)

The ring $\mathbb{Q}(n)\langle S\rangle$ possesses a field of fractions.
Field of fractions of $\mathbb{Q}(n)\langle S\rangle$ defined by:

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- Multiplication:

$$\frac{D}{C} \cdot \frac{A}{B} = \frac{VD}{VC} \cdot \frac{UA}{UB} = \frac{UA}{VC}.$$

Application to Chebyshev Recurrences

Definition

Let φ be the "Chebyshev" morphism from the differential ring into the recurrence ring: $\varphi(x) = 1/2\left(S + S^{-1}\right)$ and $\varphi\left(\frac{d}{dx}\right) = \frac{2n}{-S+S^{-1}}$.

Theorem

Let f be a function and L be a differential operator such that: $L \cdot f = 0$
A numerator of a fraction of recurrence operator $\varphi(L)$ is a Chebyshev recurrence relation of f.

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$f = \sqrt{1-x^2}$ is cancelled by the differential operator: $x + (1-x^2)\frac{d}{dx}$.

$$\varphi\left(x + (1-x^2)\frac{d}{dx}\right) = \frac{S+S^{-1}}{2} + \left(1 - \frac{S^2+2+S^{-2}}{4}\right)\frac{2n}{-S+S^{-1}}$$

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$$= \frac{\left(-S+S^{-1}\right)\left(S+S^{-1}\right)}{2(-S+S^{-1})} - \frac{(n+2)S^2 + 2n - (n-2)S^{-2}}{2\left(-S+S^{-1}\right)}$$

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The Chebyshev coefficients c_n of f, satisfy the recurrence relation:

$$(n+3)c_{n+2} - 2nc_n + (n-3)c_{n-2} = 0.$$

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Example: Normalize fraction for $\sqrt{1-x^2}$

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$$=\frac{\left(-S+S^{-1}\right)\left((n+2)S-(n-2)S^{-1}\right)}{2(-S+S^{-1})}.$$

Smaller order

$$\Rightarrow (n+2)c_{n+1}-(n-2)c_{n-1}=0.$$

III Algorithms

Lewanowicz's algorithm (1976)

Horner+Normalize at each step.
Example with $f = \sqrt{1-x^2}$

$$(1-x^2)\frac{d}{dx} + x.$$

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Example with $f = \sqrt{1-x^2}$

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\varphi(1-x^2)\varphi\left(\frac{d}{dx}\right) &= \frac{(S+S^{-1})(-S+S^{-1})}{4}\frac{2n}{-S+S^{-1}} \\
&= \frac{((n+1)S-(n-1)S^{-1})}{2}
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A recurrence verified by the Chebyshev coefficients of f is:

$$(n+2)\,c_{n+1} - (n-2)\,c_{n-1} = 0$$

Algorithms of Paszkowski (1975) and Rebillard (1998)

Observation: if $D = \varphi(\frac{d}{dx}) = \frac{2n}{-S+S^{-1}}$ then D^{-1} is a polynomial.

- INPUT :

$$L = \sum_{i=0}^{k} p_i(x) \left(\frac{d}{dx}\right)^i$$

- OUTPUT : A numerator of $\varphi(L)$

Computation with polynomials only.

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Compute $q_i(x)$ such that

$$\sum_{i=0}^{k} p_i(x) \left(\frac{d}{dx}\right)^i = \sum_{i=0}^{k} \left(\frac{d}{dx}\right)^i q_i(x).$$

$$\sum_{i=0}^{k} p_i(X) D^i = \frac{\sum_{i=0}^{k} D^{-k+i} q_i(X)}{D^{-k}}.$$

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Rebillard

$$X_k := D^{-k} X D^k.$$

$$\sum_{i=0}^{k} p_i(X) D^i = \frac{\sum_{i=0}^{k} p_i(X_k) D^{-k+i}}{D^{-k}}.$$

Our algorithm: Divide and conquer

D^{-i} is of bidegree $(2i, 2i)$.

New, fast algorithm

Step 1: Compute $q_i(x)$ such that

$$\sum_{i=0}^{k} p_i(x)\left(\frac{d}{dx}\right)^i = \sum_{i=0}^{k}\left(\frac{d}{dx}\right)^i q_i(x).$$

Step 2 : Divide and conquer

$$\sum_{i=0}^{k} D^{-k+i} q_i(X) =$$

$$D^{-\frac{k}{2}} \sum_{i=0}^{\frac{k}{2}} D^{-\frac{k}{2}+i} q_i(X) + \sum_{i=\frac{k}{2}+1}^{k} D^{-k+i} q_i(X).$$

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Theorem

If the degrees of p_i are at most k,

- *New: $O(k^\omega)$ arithmetic operations.*
- *Paszkowski and Lewanowicz algorithms : $O(k^4)$ arithmetic operations.*
- *Rebillard : $O(k^5)$ arithmetic operations.*

IV Conclusion and Future Works

Other Orthogonal Polynomial Families

Same relation to multiplication by x and differentiation for the Gegenbauer and Jacobi Polynomials: same algorithm.

```
> diffeqtorecgegenbauer(y(x)-diff(y(x),x),y(x),alpha,u(n));
```

$$(-n-2-\alpha)\,u(n)+\left(2n^2+4n+4\alpha n+4\alpha+2\alpha^2\right)u(n+1)+(n+\alpha)\,u(n+2)$$

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Next

Laguerre and Hermite polynomials, Bessel functions and other special functions.

Conclusion and Future works

Contributions:

- Use of fractions of recurrence operators.
- New algorithm.
- Maple code.
- Available in the Dynamic Dictionary of Mathematical Functions.

Perspectives:

- Recurrence in other bases (Jacobi, Legendre and Laguerre polynomials, Bessel functions)
- Numerical computation of the coefficients.