Rigorous Uniform Approximation of D-finite Functions

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I Introduction

D-finite Functions

Definition

A function $y : \mathbb{R} \to \mathbb{R}$ is D-finite if it is solution of a (homogeneous) linear differential equation with polynomial coefficients:

$$L \cdot y = a_r y^{(r)} + a_{r-1} y^{(r-1)} + \cdots + a_0 y = 0, \quad a_i \in \mathbb{Q}[x]. \tag{1}$$

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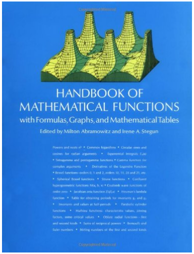
About 60% of Abramowitz & Stegun



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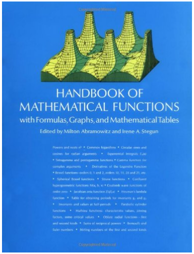

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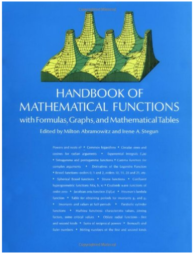
How can we approximate a D-finite function f?

Polynomial approximation:

$$f(x) \approx \sum_{i=0}^{n} f_i x^i$$



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Uniform Approximation of D-finite Functions

Problem

Given an integer d, and a D-finite function f specified by a differential equation with polynomial coefficients and suitable boundary conditions, find the coefficients of a polynomial $p(x)$ of degree d and a "small" bound R such that $|p(x) - f(x)| < R$ for all x in $[-1, 1]$.

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Applications: Repeated evaluation on a line segment

- Plot
- Numerical integration
- Computation of minimax approximation polynomials using the Remez algorithm

Rigorous Uniform Approximation of D-finite Functions

- Why?
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  - Bound roundoff, discretization, truncation errors in numerical algorithms
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- What?
  - Interval arithmetic

Recall: Basic Properties of Chebyshev Polynomials

$$T_n(cos(\theta)) = \cos(n\theta)$$

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$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}}dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = 0 \\ \frac{\pi}{2} & \text{otherwise} \end{cases}$$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \qquad (1-x^2)T_n''(x) - xT_n'(x) + n^2T_n(x)$$

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$$T_n(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k 2^{n-2k-1} \frac{n}{n-k} \binom{n-k}{k} x^{n-2k}$$

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_2(x) &= 2x^2 - 1 \\ T_3(x) &= 4x^3 - 3x \\ T_4(x) &= 8x^4 - 8x^2 + 1 \end{aligned}$$

Chebyshev Series vs Taylor Series I

Two approximations of a function f:

by Taylor series

$$f = \sum_{n=0}^{+\infty} c_n x^n,$$

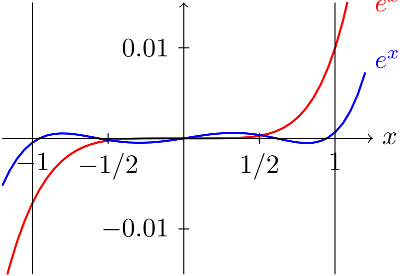
$$c_n = \frac{f^{(n)}(0)}{n!},$$

or by Chebyshev series

$$f = \sum_{n=-\infty}^{+\infty} t_n T_n(x),$$

$$t_n = \frac{1}{\pi} \int_{-1}^{1} T_n(t) \frac{f(t)}{\sqrt{1-t^2}} dt.$$

Chebyshev Series vs Taylor Series I: Example



$$e^x - (1 + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4)$$

$$\begin{aligned} e^x - (&1{,}2660T_0(x) + 1{,}1303T_1(x) \\ &+ 0{,}2714T_2(x) + 0{,}0443T_3(x) \\ &+ 0{,}0054T_4(x)) \end{aligned}$$

Chebyshev truncations are near-best

Let f be continuous on $[-1,1]$ with degree n Chebyshev truncation f_n and best approximant p_n (the polynomial of degree at most n that minimizes $\|f-p\|_\infty = \sup_{-1\leqslant x\leqslant 1}|f(x)-p(x)|$), $n\geqslant 1$. Then

$$\|f-f_n\|_\infty \leqslant \underbrace{\left(4+\frac{4}{\pi^2}\log(n+1)\right)}_{\Lambda_n}\|f-p_n\|_\infty.$$

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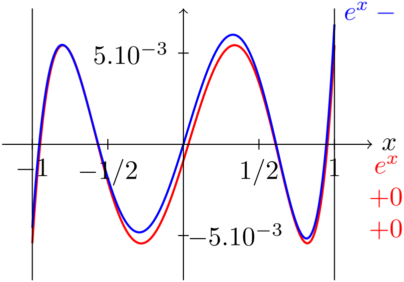
$$\|f - f_n\|_\infty \leqslant \underbrace{\left(4 + \frac{4}{\pi^2} \log(n+1)\right)}_{\Lambda_n} \|f - p_n\|_\infty.$$

- $\Lambda_{10} = 4.93...$ → we lose at most 3 bits
- $\Lambda_{30} = 5.37...$ → we lose at most 3 bits
- $\Lambda_{100} = 5.87...$ → we lose at most 3 bits
- $\Lambda_{1000} = 6.80...$ → we lose at most 3 bits

Chebyshev truncations are near-best : Example



$$e^x - (1{,}2660T_0(x) + 1{,}1303T_1(x) + 0{,}2714T_2(x) + 0{,}0443T_3(x) + 0{,}0054T_4(x))$$

$$e^x - (1.0000 + 0.9973x + 0.4988x^2 + 0.1773x^3 + 0.0441x^4)$$

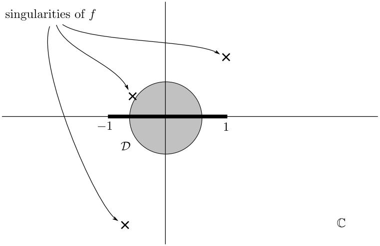
Chebyshev truncation of degree 4

Best approximant of degree 4

Chebyshev Series vs Taylor Series II

Convergence Domains :

For Taylor series:
disc centered at $x_0 = 0$ which
avoids all the singularities of f



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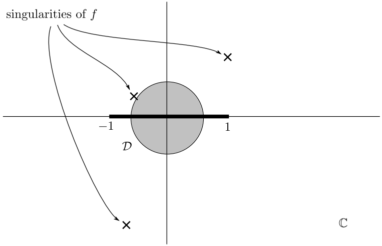

- Taylor series can not converge over entire $[-1,1]$ unless all singularities lie outside the unit circle.

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For Chebyshev series:
elliptic disc with foci at ± 1 which
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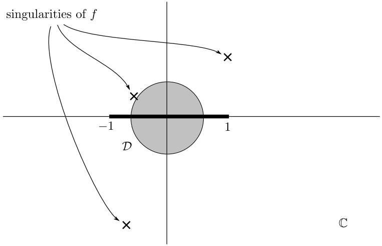

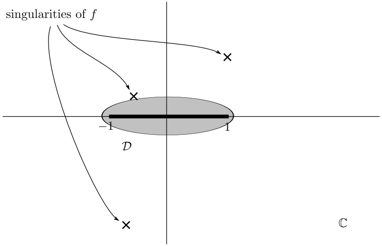

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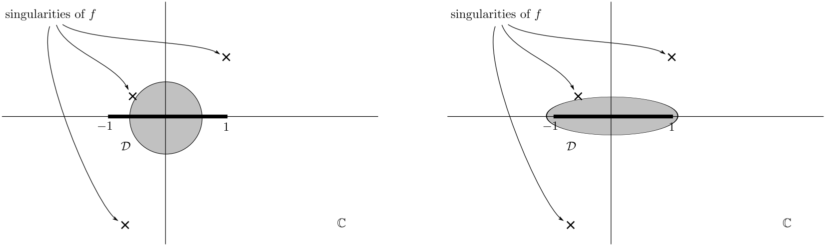

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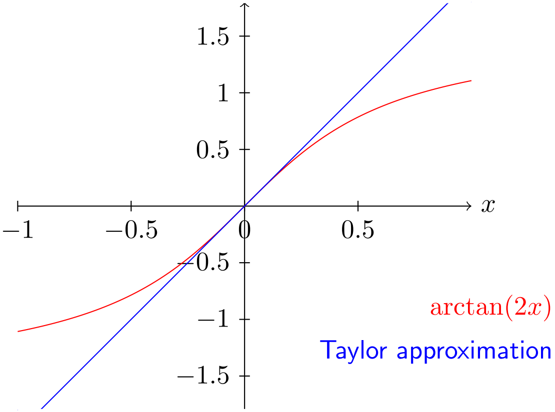
✓ Chebyshev series converge over entire $[-1,1]$ as soon as there are no real singularities in $[-1,1]$.

Chebyshev Series vs Taylor Series II: Example

Approximation of $\arctan(2x)$ by Taylor expansion of degree 1



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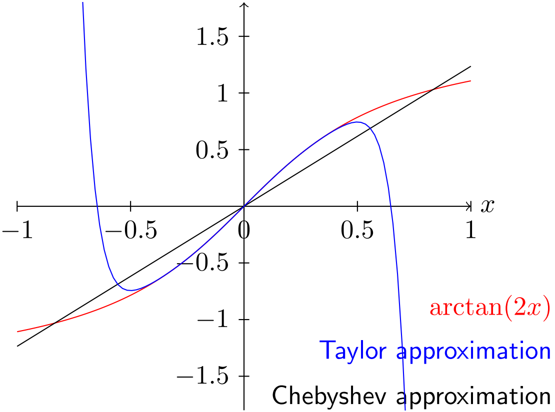
Approximation of $\arctan(2x)$ by Chebyshev expansion of degree 1



Previous Work

- Computation of the Chebyshev coefficients for D-finite functions
  - Lánczos (1938) – τ method
  - Clenshaw (1957) – iterative computation like Miller
- Recurrence relation
  - Fox & Parker (1968) – small orders, link with Clenshaw
  - Paszkowski (1975) – general case
  - Geddes (1977), Rebillard (1998), Benoit & Salvy (2009) – symbolic computation
- Interval arithmetic using truncated Chebyshev expansions
  - Kaucher & Miranker (1984) – ultra-arithmetic
  - Brisebarre & Joldeș (2010) – ChebModels

Our Work

Given a linear differential equation with polynomial coefficients, boundary conditions and an integer d

- Compute a polynomial approximation p on $[-1,1]$ of degree d of the solution f in the Chebyshev basis in $O(d)$ arithmetic operations.
- Compute a sharp bound R such that $|f(x) - p(x)| < R$, $x \in [-1,1]$ in $O(d)$ arithmetic operations.

II Computation of the coefficients

Chebyshev Series of D-finite Functions

Theorem (60's, BenoitJoldesMezzarobba12)

$\sum u_n T_n(x)$ *is solution of a linear differential equation with polynomial coefficients iff the sequence* u_n *is cancelled by a linear recurrence with polynomial coefficients.*

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Recurrence relation + good initial conditions $\Rightarrow$ Fast numerical computation of the coefficients

Taylor: $\exp = \sum \frac{1}{n!} x^n$
Rec: $u_{n+1} = \frac{u_n}{n+1}$

$u_0 = 1$	$1/0! = 1$
$u_1 = 1$	$1/1! = 1$
$u_2 = 0{,}5$	$1/2! = 0{,}5$
$\vdots$	$\vdots$
$u_{50} \approx 3{,}28.10^{-65}$	$1/50! \approx 3{,}28.10^{-65}$

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Chebyshev: $\exp = \sum \mathrm{I}_n(1) T_n(x)$
Rec: $u_{n+1} = -2n u_n + u_{n-1}$

$u_0 = 1{,}266$	$\mathrm{I}_0(1) \approx 1{,}266$
$u_1 = 0{,}565$	$\mathrm{I}_1(1) \approx 0{,}565$
$u_2 \approx 0{,}136$	$\mathrm{I}_2(1) \approx 0{,}136$
$\vdots$	$\vdots$

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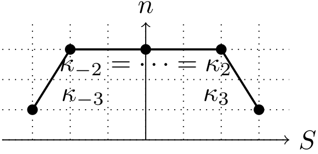
$\vdots$	$\vdots$
$u_{50} \approx 4{,}450.10^{67}$	$\mathrm{I}_{50}(1) \approx 2{,}934.10^{-80}$

Convergent and Divergent Solutions of the Recurrence

Study of the Chebyshev recurrence

If u_n is solution, then there exists another solution $v_n \sim \frac{1}{u_n}$

Newton polygon of a Chebyshev recurrence



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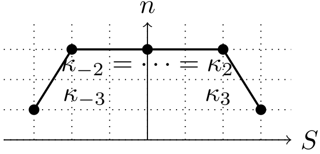

For the recurrence $u_{n+1} + 2nu_n - u_{n-1}$

Two independent solutions are $\mathrm{I}_n(1) \sim \frac{1}{(2n)!}$ and $K_n(1) \sim (2n)!$

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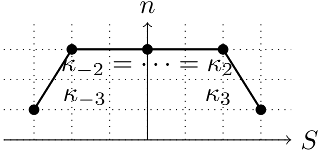

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Miller's algorithm

To compute the first N coefficients of the most convergent solution of a recurrence relation of order 2

- Initialize $u_N = 0$ and $u_{N-1} = 1$ and compute the first coefficients using the recurrence backwards
- Normalize u with the initial condition of the recurrence

Back to exp

Example

$y(x) = e^x = \sum_{n=-\infty}^{\infty} c_n \, T_n(x)$ $\qquad$ $c_{n+1} + 2n \, c_n - c_{n-1} = 0$

u_0	c_0
u_1	c_1
u_2	c_2
$\vdots$	$\vdots$
u_{50}	c_{50}
u_{51}	c_{51}
u_{52}	c_{52}

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$u_{51} = \quad 1$	c_{51}
$u_{52} = \quad 0$	c_{52}

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$$\begin{aligned}
&u_0 \\
&u_1 \\
&u_2 \\
&\vdots \\
&u_{50} \approx 1{,}02 \cdot 10^2 \\
&u_{51} = 1 \\
&u_{52} = 0
\end{aligned}
\qquad\qquad
\begin{aligned}
&c_0 \\
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$$\begin{aligned}
u_0 & \\
u_1 & \\
u_2 &\approx -4{,}72 \cdot 10^{80} \\
&\vdots \\
u_{50} &\approx \quad 1{,}02 \cdot 10^{2} \\
u_{51} &= \quad 1 \\
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$$
\begin{array}{lll}
u_0 & & \\
u_1 & \approx & 1{,}96 \cdot 10^{81} \\
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$$\begin{array}{c} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_{50} \\ c_{51} \\ c_{52} \end{array}$$

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c_0 \\
c_1 \\
c_2 \\
\vdots \\
c_{50} \\
c_{51} \\
c_{52}
\end{array}$$

$$S = \sum_{n=-50}^{50} u_n \, T_n(0) \approx -3{,}48 \cdot 10^{81}$$

Back to $\exp$

Example

$y(x) = e^x = \sum_{n=-\infty}^{\infty} c_n \, T_n(x)$ $\qquad c_{n+1} + 2n\, c_n - c_{n-1} = 0$

$$\begin{array}{lcl} u_0 & \approx & -4{,}40 \cdot 10^{81} \\ u_1 & \approx & 1{,}96 \cdot 10^{81} \\ u_2 & \approx & -4{,}72 \cdot 10^{80} \\ \vdots & & \\ u_{50} & \approx & 1{,}02 \cdot 10^{2} \\ u_{51} & = & 1 \\ u_{52} & = & 0 \end{array} \quad \xrightarrow{c_n := u_n / S} \quad \begin{array}{lcl} c_0 & \approx & 1{,}27 \\ c_1 & \approx & -5{,}65 \cdot 10^{-1} \\ c_2 & \approx & 1{,}36 \cdot 10^{-1} \\ \vdots & & \\ c_{50} & \approx & 2{,}93 \cdot 10^{-80} \\ c_{51} & \approx & 2{,}88 \cdot 10^{-82} \\ c_{52} & \approx & 0 \end{array}$$

$$S = \sum_{n=-50}^{50} u_n \, T_n(0) \approx -3{,}48 \cdot 10^{81}$$

Algorithm for Computing the Coefficients

Algorithm

Input: *a differential equation of order r with boundary conditions*
Output: *a polynomial approximation of degree N of the solution*

- *compute the Chebyshev recurrence of order $2s \geqslant 2r$*
- *for i from 1 to s*
  - *using the recurrence relation backwards, compute the first N coefficients of the sequence $u^{[i]}$ starting with the initial conditions*

$$\left(u^{[i]}(N+2s),\cdots,u^{[i]}(N+i),\cdots,u^{[i]}(N+1)\right)=(0,\cdots,1,\cdots,0)$$

- *combine the s sequences $u^{[i]}$ according to the r boundary conditions and the $s-r$ symmetry relations*

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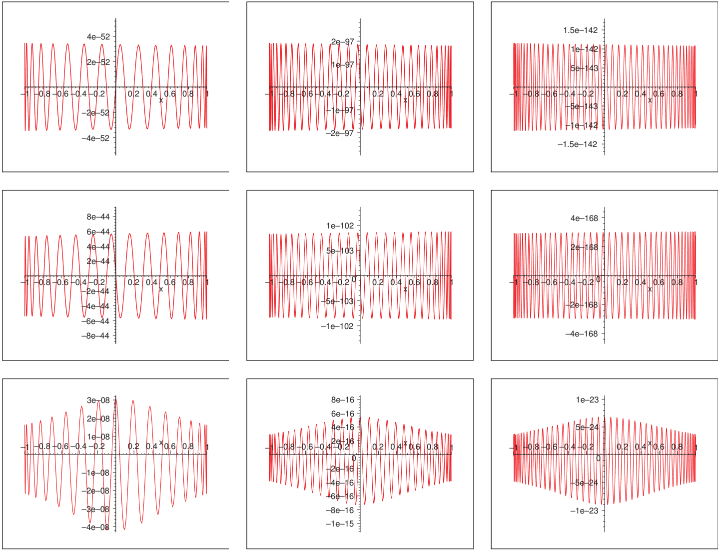
Theorem

This algorithm runs in $O(N)$ arithmetic operations

Quality of polynomial approximations



$\frac{e^{x/2}}{\sqrt{x+16}}$

$\frac{3\cos x - \sin x}{2}$

$e^{1/(1+2x^2)}$

degree = 30 degree = 60 degree = 90

III Validation

Our Work

Given a linear differential equation with polynomial coefficients, boundary conditions and an integer d

- ✓ Compute a polynomial approximation p on $[-1,1]$ of degree d of the solution f in the Chebyshev basis in $O(d)$ arithmetic operations.
- Compute a sharp bound R such that $|f(x) - p(x)| < R$, $x \in [-1,1]$ in $O(d)$ arithmetic operations.

Fixed Point Theorem Applied to a Differential Equation

f is solution of

$$y'(x) - a(x)y(x) = 0, \text{ with } y(0) = y_0,$$

if and only if f is a fixed point of τ defined by

$$\tau(y)(t) = y_0 + \int_0^t a(x)y(x)dx.$$

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Given p, find i, p_i and R_i such that $\tau^i(B(p,R)) \subset B(p_i,R_i) \subset B(p,R)$

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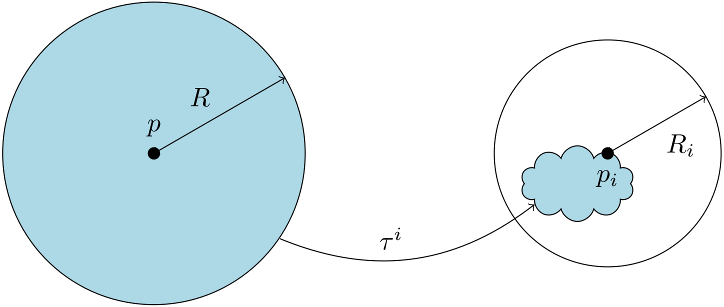

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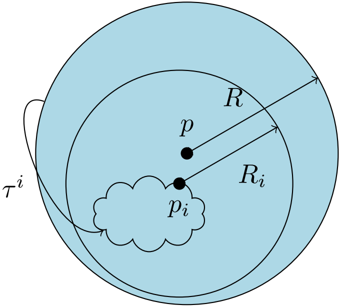

$$f \in B(p,R) \implies f = \tau^i(f) \in B(p_i, R_i)$$

$$\|f-p\|_\infty \leqslant \|f-p_i\|_\infty + \|p-p_i\|_\infty \leqslant R_i + \|p-p_i\|_\infty \leqslant R$$

Algorithm for a Differential Equation of Order 1

Given p and τ find i, p_i and R_i such that $\tau^i(B(p,R)) \subset B(p_i,R_i) \subset B(p,R)$

Algorithm (Find R)

- $p_0 := p$
- *while* $i! < \|a\|_\infty^i$
    - *Compute* $p_i(t)$ *a rigorous approximation of* $y_0 + \int_0^t a(x)p_{i-1}(x)dx$
    - $M_i = \|\tau(p_{i-1}) - p_i\|_\infty$
- *Return*

$$R = \frac{\|p_i - p\|_\infty + \sum_{j=1}^{i} M_j \frac{\|a\|_\infty^j}{j!}}{1 - \frac{\|a\|_\infty^i}{i!}}$$

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$$R_i = \sum_{j=1}^{i} \frac{\|a\|_\infty^j M_j}{j!} + \|a\|_\infty^i R \frac{|t|^i}{i!}$$

General Algorithm

Algorithm

INPUT: a differential equation $L = y^{(r)} + a_{r-1}y^{(r-1)} + \cdots + a_0 y$ *and initial conditions* $y^{r-1}, \ldots, y^0$ *and a polynomial* p

OUTPUT: a real $R > 0$ *such that* $\|y - p\|_\infty < R$

- $p_0^{(0)} := p$; $p_0^{(1)} = p'$; $\ldots$; $p_0^{(r-1)} = p^{(r-1)}$
- *Compute* $A \geqslant \max_{k=0}^{r-1} \|a_k\|_\infty$
- *for* $i = 0, 1, \ldots$ *while* $\gamma_i := \frac{A^i e}{i!} \leqslant \frac{1}{2}$
    - $p_{i+1}^{(r-1)} := y^{(r-1)}(0) + \int_0^t \left(a_{r-1}p_i^{(r-1)} + a_{r-2}p_i^{(r-2)} + \cdots + a_0 p_i^{(0)}\right)$
    - *for* k *from* $r-2$ *to* 0
        - *Compute* $p_{i+1}^{(k)} := y^{(k)}(0) + \int_0^x p_{i+1}^{(k+1)}$
- $\alpha := re^{A+1}\epsilon$
- *Compute* $\beta_i \leqslant \left\|p^{(r-1)} - p_i^{(r-1)}\right\|_\infty$
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IV Conclusion

Our Work

Given a linear differential equation with polynomial coefficients, boundary conditions and an integer d

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- ✓ Compute a sharp bound R such that $|f(x) - p(x)| < R$, $x \in [-1,1]$ in $O(d)$ arithmetic operations.

Final Algorithm

Algorithm

INPUT: Differential equation with boundary conditions and a degree d
OUTPUT: a polynomial approximation of degree d *and a bound*

- *Compute an approximation* P *of degree* d *of the solution with the first algorithm*
- *Compute the bound* R *of the approximation with the second algorithm*
- *return the pair* P, R

Quality of Bounds

$$\log_{10} \frac{\text{(bounds computed)}}{\|y-p\|_\infty}$$

$\frac{e^{x/2}}{\sqrt{x+16}}$	4,8	0,58	0, 57
$\frac{3\cos x - \sin x}{2}$	3,1	3,7	4,1
$e^{1/(1+2x^2)}$	0,57	0,56	0,56
	degree = 30	degree = 60	degree = 90

Random Example

Example

$(x+5)y^{(3)}(x)+(-x^3-5x^2+4x+5)y^{(2)}(x)+(6x^3+6+3x)y^{(1)}(x)+(-3x^3-x^2-2x+4)y(x)=0, \quad y(0)=-6, y^{(1)}(0)=1, y^{(2)}(0)=-2$

Compute coefficients of polynomial of degree 30.
Validated bound: $0.58 \cdot 10^{-14}$.

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- What about other expansions (Gegenbauer polynomials, Hermite polynomials, Laguerre polynomials)?
- What about the solution of non linear differential equation (ex: $\tan(x)$)? Newton method instead of use of recurrence relation